

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

PROBLEM SHEET 1

MA 4010 : Real Analysis

Note: Euclidean metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$; Euclidean metric on $\mathbb{R}^2$ is given by $d(\bar{x}, \bar{y}) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ for $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2) \in \mathbb{R}^2$.

1. Let E be a non empty subset of an ordered set. Suppose α is lower bound for A and β is upper bound for A . Then prove that $\alpha \leq \beta$.
2. Let $\emptyset \neq A$ be a subset of $\mathbb{R}$ which bounded below. Prove that $\inf A = -\sup \{-A\}$.
3. Prove that no order can be defined on $\mathbb{C}$ that turns it into an ordered field.
4. Determine which of the following ordered sets are having lub property?
 - (a) $A_1 = [2, 3]$ order on A_1 is the order induced by $(\mathbb{R}, <)$.
 - (b) Define $A_2 = \mathbb{R} - \{\frac{n-1}{n} : n \in \mathbb{N}\}$. A order on A_2 is the induced order from $(\mathbb{R}, <)$.
 - (c) For $z = a + ib, w = c + id \in \mathbb{C}$, define order on $\mathbb{C}$ by $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that $A_3 = (\mathbb{C}, <)$ is an ordered set.
 - (d) $A_4 = [1, 3) \cup (4, 5]$. The order $<$ on A_4 is given by for $x, y \in A_4$, we say $x < y$ if $y < x$ in $(\mathbb{R}, <)$.
5. Under what condition does the equality holds in the Schwartz inequality?
6. Prove or disprove:
 - (a) $f : A \rightarrow \mathbb{N}$ is one to one. Then A is countable.
 - (b) There exist an uncountable set A and a function f such that $f : Q \rightarrow A$ is onto.
7. Let A be any non-empty set and $B \subsetneq A$ and there exists $f : B \rightarrow A$ which is bijection. Prove that A is infinite.
8. Find out the limit points of the following sets
 - (a) $E_1 = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}^c\}$.
 - (b) $E_2 = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1 \text{ and } |y| > 3\}$.
9. Determine the compactness of the following sets?
 - (a) $F_1 = \bigcap_{n \in \mathbb{N}} [1, 1 + \frac{1}{n})$.
 - (b) $F_2 = A$ subset of real numbers which is infinite such that for any two elements in the set having distance at least one.
10. Prove or disprove which of the following d_i 's are metric on $\mathbb{R}$. For $x, y \in \mathbb{R}$,
 - (a) $d_1(x, y) = (x - y)^2$.
 - (b) $d_2(x, y) = \frac{|x-y|}{1+|x-y|}$.

- (c) $d_3(x, y) = \min \{|x - y|, 1\}$.
11. For the metric space $(\mathbb{R}, d_3)$ given in problem (10) prove that open sets in d_3 metric are open in Euclidean metric on $\mathbb{R}$. What can you say about the converse? Whether compact sets in $(\mathbb{R}, d_3)$ is also compact in Euclidean metric on $\mathbb{R}$? What can you say about the converse?
 12. Find out the closures and interiors of the following sets.
 - (a) $A_1 = (0, 1] \cap \mathbb{Q}$.
 - (b) $A_2 = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < 1\} \cup \{(x, y) \in \mathbb{R}^2 : |x - 2|^2 + |y|^2 < 1\}$.
 - (c) $A_3 = A_2 \cup \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.
 13. Prove that every connected metric space with at least two points is uncountable.
 14. Are closures and interiors of a connected set always connected?
 15. Is there a non-empty perfect set in $\mathbb{R}$ which contains no rational number?
 16. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E compact? Is E perfect?